PRACTICAL CONSIDERATIONS FOR THERMAL STRESSES INDUCED BY SURFACE HEATING

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ABSTRACT

Rapid surface heating can induce large stresses in solids. A relatively simple model, assuming full constraint in two dimensions and no constraint in the third dimension, can adequately model stresses in a wide variety of situations. This paper derives this simple model, and supports it with criteria for its validity. Phenomena that are considered include non-zero penetration depths for the heat deposition, spatial non-uniformity in the surface heating, and elastic waves. Models for each of these cases, using simplified geometries, are used to develop quantitative limits for their applicability.

I. INTRODUCTION

Rapid surface heating can induce large stresses in solids, possibly leading to surface roughening, yielding, or fracture. The determination of the stresses for a given material and set of loads can be quite difficult, requiring a time-dependent, three dimensional analysis. For many cases, though, a relatively simple model, assuming full constraint in two dimensions and no constraint in the third dimension, can adequately model the peak stress. This paper derives such a model, and supports it with criteria for its validity. Phenomena that are considered include non-zero penetration depths for the heat deposition, spatial non-uniformity in the surface heating, and elastic waves. Models for each of these cases, using simplified geometries, is used to develop quantitative limits for their applicability. Thermal waves are an additional phenomenon that can be of concern for very short pulses, but this effect is left for future work.

II. BASELINE CASE

The base case considers a solid restrained from deformation in two dimensions and without constraint in the third dimension. The lack of constraint in the third dimension is a result of the free surface. The full constraint in the other dimensions assumes that the thermal field is shallow relative to the depth of the structure, so that cold material below the surface restrains motion of the heated material in that shallow layer. To develop a model for stresses in this situation, we begin with the stress-strain relations:

$$\begin{aligned}\sigma_{xx} &= \lambda(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})+2\mu\varepsilon_{xx}-(3\lambda+2\mu)\alpha T\\ \sigma_{yy} &= \lambda(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})+2\mu\varepsilon_{yy}-(3\lambda+2\mu)\alpha T\\ \sigma_{zz} &= \lambda(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})+2\mu\varepsilon_{zz}-(3\lambda+2\mu)\alpha T\end{aligned} \quad (1)$$

where λ is the Lame constant, μ is the shear modulus, and α is the thermal expansion coefficient. This equation assumes that T is the temperature difference from a stress-free reference temperature. Under the assumptions discussed above, we impose

$$\varepsilon_{zz}=\varepsilon_{yy}=\sigma_{xx}=0 \quad (2)$$

From the last of these conditions, combined with Eq. 1, we find the strain in the x direction to be

$$\varepsilon_{xx}=\left(\frac{3\lambda+2\mu}{\lambda+2\mu}\right)\alpha T \quad (3)$$

This can then be substituted into Eq. 1 to find the transverse stresses, which can be written as:

$$\sigma_{yy}=\sigma_{zz}=\frac{-E\alpha T}{1-\nu} \quad (4)$$

where E is the elastic modulus and ν is Poisson's ratio. To complete the model we need an estimate of the surface temperature induced by the surface heating. Assuming uniform heating applied on a half-space, the surface temperature is[1]

$$T_{surrface}=\frac{2q}{k}\sqrt{\frac{\kappa t}{\pi}} \quad (5)$$

where q is the surface heat flux, k is the thermal conductivity, and κ is the thermal diffusivity. Combining

Eqs. 4 and 5 provides the following model for stresses induced by rapid surface heating:

$$\sigma_{yy} = \sigma_{zz} = \frac{-2qE\alpha}{(1-\nu)k}\sqrt{\frac{\kappa t}{\pi}} \qquad (6)$$

$$\sigma_{xx} = 0$$

This result provides a baseline estimation of the surface stresses induced by rapid surface heating. It assumes spatial uniformity of the applied heat, no volumetric heating below the surface, and it ignores both elastic and thermal waves.

III. DEPOSITION BELOW THE SURFACE

Most surface heating actually deposits heat as volumetric heating within a thin layer near the surface. A typical model for volumetric heating resulting from energy impinging on a surface is

$$Q''' = Ae^{-\gamma x} \qquad (7)$$

Where A is a constant and γ is the attenuation coefficient. To provide the same total heat input as a true surface heating flux q, we must enforce $A=q\gamma$. The temperature distribution resulting from volumetric heating of this type is[1]

$$T = \frac{2q}{k\gamma}\left[\zeta \; ierfc\left(\frac{\eta}{2\zeta}\right) - e^{-\eta} + e^{\zeta^2-\eta} erfc\left(\zeta - \frac{\eta}{2\zeta}\right) + e^{\zeta^2+\eta} erfc\left(\zeta + \frac{\eta}{2\zeta}\right)\right] \qquad (8)$$

where $\zeta = \gamma\sqrt{\kappa t}$, representing the ratio of the diffusion length in time t to the characteristic deposition length, and $\eta = x\gamma$. The surface temperature resulting from this solution is

$$T_{surface} = \frac{q}{k\gamma}\left[\frac{2\zeta}{\sqrt{\pi}} - 1 + e^{\zeta^2} erfc(\zeta)\right] \qquad (9)$$

The ratio of the surface temperature from Eq. 9 to the surface temperature due to surface heating (Eq. 5) is

$$R = 1 - \frac{\sqrt{\pi}}{2\zeta}\left[1 - e^{\zeta^2} erfc(\zeta)\right] \qquad (10)$$

Fig. 1 provides a plot of this ratio as a function of ζ. The corresponding stresses would follow the same curve. From this curve one can see that the effect is less than 10% for $\zeta>8$ and less than 1% for $\zeta>60$. This latter result was determined using the asymptotic result:

$$R \sim 1 - \frac{\sqrt{\pi}}{2\zeta} + \frac{1}{2\zeta^2} \qquad (11)$$

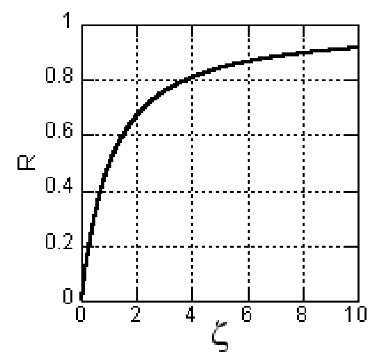

Figure 1: Ratio of surface temperatures due to volumetric heating and equivalent surface heating

IV. Spatial Non-Uniformity in Surface Heating

Quite often the heating distribution over the surface is non-uniform. For example, many lasers produce a gaussian heating distribution when the laser is normally incident on a flat surface. To explore this effect, we consider the solution by Hector and Hetnarski[2]. This gives the peak stress due to laser heating with a gaussian shape on a half-space as (at r*=z*=0):

$$\sigma_{rr}^* = \int_0^\infty h(\beta^*)\int_0^{t^*}\left[\left(\frac{G^*}{2}-1\right)\frac{\beta^{*2}}{2} + \left(\beta^{*2} - \frac{1}{2}\frac{\partial^2 G^*}{\partial z^{*2}}\right) + \left(\frac{1}{2}+\nu\right)\beta^{*2} erfc\left(\frac{\beta^*\sqrt{t^*-\tau^*}}{2}\right)\right] d\tau^* d\beta^* \qquad (12)$$

where

$$G^* = 2\,erf\left(\frac{\beta^*}{2}\sqrt{t^*-\tau^*}\right) \qquad (13)$$

$$\frac{\partial^2 G^*}{\partial z^{*2}} = \beta^* \left[\frac{4}{\sqrt{\pi(t^* - \tau^*)}} \exp\left\{ \frac{-\beta^{*2}}{4} (t^* - \tau^*) \right\} + 2\beta^* \, erf\left(\frac{\beta^*}{2} \sqrt{t^* - \tau^*} \right) \right] \quad (14)$$

and

$$h(\beta^*) = \exp\left(\frac{-\beta^{*2}}{4} \right). \quad (15)$$

In these equations, the starred quantities are all dimensionless, according to:

$$\beta^* = \frac{\beta}{\sqrt{K_c}} \qquad \sigma_{ij}^* = \frac{4(1-\nu)k\sqrt{K_c}}{(1+\nu)\alpha\mu q_0}\sigma_{ij}$$
$$t^* = 4\kappa K_c t \quad (16)$$
$$\tau^* = 4\kappa K_c \tau \qquad T^* = \frac{k\sqrt{K_c}}{q_0} T$$

Here τ and β are integration variables, K_c is a measure of the width of the gaussian laser profile on the surface, and q_0 is the peak surface heating. Putting Eqs. 13-15 together gives

$$\sigma_{rr}^* = \int_0^\infty \beta^* \exp\left(\frac{-\beta^{*2}}{4} \right)^* \times \int_0^{t^*} \left[\beta^* (1+\nu) erfc\left(\frac{\beta^*}{2} \sqrt{t^* - \tau^*} \right) - \frac{2}{\sqrt{\pi(t^* - \tau^*)}} \exp\left\{ \frac{-\beta^{*2}}{4} (t^* - \tau^*) \right\} \right] d\tau^* d\beta^* \quad (17)$$

Carrying out this integration provides:

$$\sigma_{rr}^* = 2(1+\nu)\sqrt{\pi}\, t^* - 4(1+\nu)\sqrt{\frac{t^*}{\pi}} - \frac{4}{\sqrt{\pi}} \left[1 + t^* - (1-t^*)\nu \right] \tan^{-1}\left(\sqrt{t^*} \right) \quad (18)$$

To compare this to our simple analytical solution, we can write the stress from Eq. 6 using the dimensionless variables in Eq. 16, giving

$$\sigma_{base}^* = -8\sqrt{\frac{t^*}{\pi}} \quad (19)$$

Hence, the ratio of the stress due to a gaussian heating profile to that of the uniform heating profile is

$$R_\sigma = \frac{-(1+\nu)}{4}\pi\sqrt{t^*} + \frac{(1+\nu)}{2} + \frac{1}{2}\left[1 + t^* - (1-t^*)\nu \right] \frac{\tan^{-1}\left(\sqrt{t^*} \right)}{\sqrt{t^*}} \quad (20)$$

This ratio is plotted in Fig. 2 as a function of the dimensionless time for several values of the Poisson ratio.

A similar approach can be taken with the temperature. Hector and Hetnarski[2] give the temperature as

$$T^* = \frac{1}{\sqrt{4\pi}} \int_0^{t^*} \frac{d\tau^*}{(1 + t^* - \tau^*)\sqrt{t^* - \tau^*}} \quad (21)$$

Carrying out this integral gives

$$T^* = \frac{\tan^{-1}\left(\sqrt{t^*} \right)}{\sqrt{\pi}} \quad (22)$$

In the dimensionless units given in Eq. 16, the one-dimensional surface temperature from Eq. 5 becomes

$$T_{base}^* = \sqrt{\frac{t^*}{\pi}} \quad (23)$$

Hence, the ratio of the temperature due to the gaussian heating profile to that of the uniform profile is

$$R_T = \frac{\tan^{-1}\left(\sqrt{t^*} \right)}{\sqrt{t^*}} \quad (24)$$

This ratio is plotted in Fig. 3.

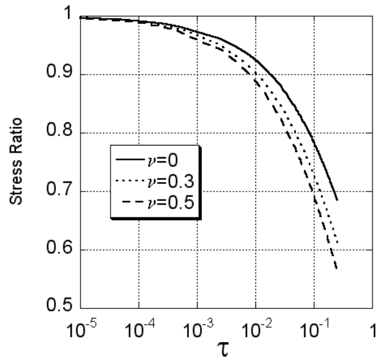

Figure 2: Ratio of stresses due to gaussian and uniform heating profiles

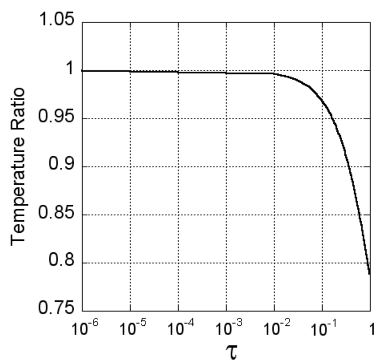

Figure 3: Ratio of stresses due to gaussian and uniform heating profiles

V. Elastic Waves

To model elastic waves, we must include inertial terms in the stress equations. To estimate their effects, consider thermoelastic deformation of a half-space, with x denoting the perpendicular distance from the surface. Following Sternberg and Chakravorty[3], one can define the following dimensionless variables

$$\xi = \frac{x}{a} \qquad \phi = \frac{kT}{qa}$$
$$\tau_w = \frac{\kappa t}{a^2} \quad \hat{\sigma}_x = \frac{(1-2\nu)}{2(1+\nu)}\frac{k\sigma_x}{\alpha\, qa\mu} \tag{25}$$
$$a = \frac{\kappa}{c} \quad c^2 = \frac{2(1-\nu)\mu}{(1-2\nu)\rho}$$

Here c is the wave speed and ρ is the density of the solid. With these definitions, the governing equations then become:

$$\frac{\partial^2\phi}{\partial\xi^2} = \frac{\partial\phi}{\partial\tau_w}$$
$$\frac{\partial^2\hat{\sigma}_x}{\partial\xi^2} = \frac{\partial^2\hat{\sigma}_x}{\partial\tau_w^2} + \frac{\partial^2\phi}{\partial\tau_w^2} \tag{26}$$

It is assumed here that the only non-zero displacement is perpendicular to the surface of the half-space. That is, $u_y=u_z=0$.The initial conditions are such that all temperatures, stresses, and time derivatives are nonexistent. The boundary conditions are that the temperatures and stresses vanish at x equals infinity, while at the surface

$$\frac{\partial\phi}{\partial\xi} = -1$$
$$\hat{\sigma}_x = 0 \tag{27}$$

The solution for the dimensionless temperature is given by[1]

$$\phi = 2\left(\sqrt{\frac{\tau_w}{\pi}}\exp\left[\frac{-\xi^2}{4\tau_w}\right] - \frac{\xi}{2}\, erfc\left[\frac{\xi}{2\sqrt{\tau_w}}\right]\right) \tag{28}$$

and the stresses can be found to be

$$\hat{\sigma}_x = -\frac{1}{2}\exp(\tau_w - \xi)\left\{1 - \exp(2\xi)\, erfc\left(\frac{2\tau+\xi}{2\sqrt{\tau_w}}\right)\right.$$
$$\left. + erf\left(\frac{2\tau_w - \xi}{2\sqrt{\tau_w}}\right) - 2\ erf\left(\sqrt{\tau_w - \xi}\right)H(\tau_w - \xi)\right\} \tag{29}$$

and

$$\hat{\sigma}_y = \hat{\sigma}_z = \nu\,\hat{\sigma}_x - (1-2\nu)\,\phi \tag{30}$$

This completes our solution for the stresses induced by surface heating on a half-space.

Since the longitudinal stress ($\hat{\sigma}_x$) is zero at the surface, the transverse stress ($\hat{\sigma}_y$) at the surface is given by

$$\hat{\sigma}_y = \hat{\sigma}_z = -(1-2\nu)\,\phi \tag{31}$$

or

$$\hat{\sigma}_y = \hat{\sigma}_z = -2(1-2\nu)\left(\sqrt{\frac{\tau_w}{\pi}}\right) \tag{32}$$

The peak stress in the wave occurs at $\xi=\tau_w$. Substituting this into Eqs. (29) and (30) gives

$$\begin{aligned}
\hat{\sigma}_x &= -\frac{1}{2}\left\{1-\exp(2\tau_w)\, erfc\left(\frac{3\sqrt{\tau_w}}{2}\right) + erf\left(\frac{\sqrt{\tau_w}}{2}\right)\right\} \\
\hat{\sigma}_y &= -\frac{\nu}{2}\left\{1-\exp(2\tau_w)\, erfc\left(\frac{3\sqrt{\tau_w}}{2}\right) + erf\left(\frac{\sqrt{\tau_w}}{2}\right)\right\} \\
&\quad -2(1-2\nu)\left(\sqrt{\frac{\tau_w}{\pi}}\exp\left[\frac{-\tau_w}{4}\right] - \frac{\tau_w}{2}\, erfc\left[\frac{\sqrt{\tau_w}}{2}\right]\right)
\end{aligned} \tag{33}$$

For long times the longitudinal stress approaches –1, while the transverse stress approaches ($-\nu$).

Typical wave shapes are shown in Fig. 4, which plots the two dimensionless stresses as a function of distance from the surface. These results are given for dimensionless times of 0.5, 1, and 10. As one would expect, the stresses are all compressive, and the peak stress in the wave occurs at $\xi=\tau_w$. Except at early times, the transverse stress peaks at the surface because that's where the temperature peaks. At early times, there is a local peak in the transverse stress where the wave front lies, and at this point the transverse stress is less than the longitudinal stress.

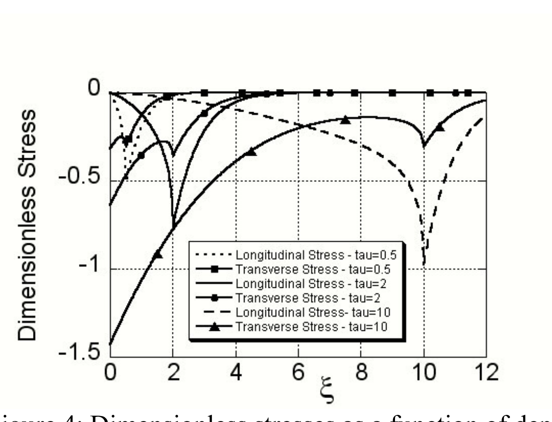

Figure 4: Dimensionless stresses as a function of depth for different times.

The peak in the longitudinal stress occurs at $\xi=\tau_w$, while the peak transverse stress occurs at the surface (except at short times). These peaks are plotted in Fig 5, which gives both stresses at $\xi=\tau_w$ along with the surface stress at the same dimensionless time. It can be seen that beyond a dimensionless time of approximately 4, the surface stress exceeds the stress at the wave peak.

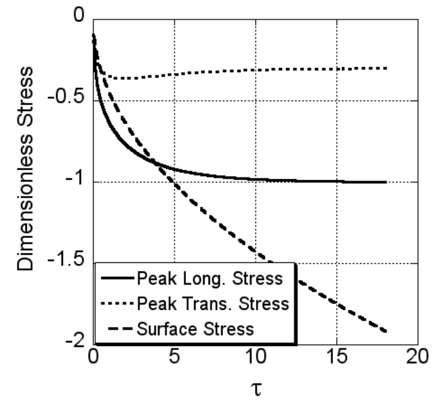

Figure 5: Stresses at surface and at wave front

The ratio of the longitudinal stress at the wave peak to the surface stress is given by:

$$R_w = \frac{1-\exp(2\tau_w)\, erfc\left(\frac{3\sqrt{\tau_w}}{2}\right) + erf\left(\frac{\sqrt{\tau_w}}{2}\right)}{4(1-2\nu)\left(\sqrt{\frac{\tau_w}{\pi}}\right)} \tag{34}$$

This ratio is plotted in Fig. 6. For large times, this ratio is 0. As τ_w approaches 0, this ratio approaches $1.10/(1-2\nu)$.

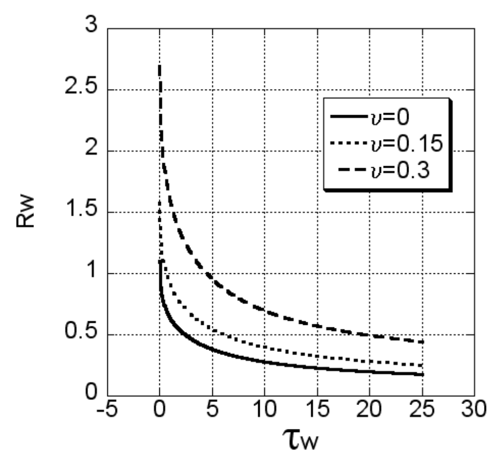

Figure 6: Ratio of peak longitudinal stress to surface stress

VI. Conclusions

For most situations, a simple formula provides an adequate representation of the thermal stress induced in a rapidly heated solid. When this formula is not valid, there are often simple analytical representations of these stresses. This paper provides these formulas, along with their regions of validity.

Acknowledgements

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References

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