Elastic Waves Induced by Surface Heating in a Half-Space

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Abstract

Rapid surface heating will induce waves in an elastic material. Closed form solutions for the resulting longitudinal and transverse thermal stresses are derived using Laplace Transforms. The model is one-dimensional, consisting of a half-space subjected to a step change in the surface heating. The transverse stress at the wave peak is found to exceed the surface stress for short times, while for long times the surface stress far exceeds either of the stresses at the wave peak. Both the longitudinal and transverse stresses at the peak of the wave reach steady state values after a few dimensionless times.

Introduction

Rapid surface heating, such as that created by a laser, can induce numerous phenomena in solids. Some of these include melting, vaporization, thermal waves [1], and plastic deformation. In many applications, such as mirrors, such phenomena must be avoided in order to ensure a long life. In this paper, I derive a closed-form solution for one-dimensional elastic waves induced by a step change in surface heating. This creates a temperature field in which the surface temperature increases as the square root of the pulse time. It is assumed that the heat is deposited at the surface, there is no cooling, the heat transport is diffusional, and that the elastic and thermal equations are uncoupled.

Several previous works have analyzed thermoelastic waves in solids. Sternberg and Chakravorty [2] solved the problem for both a step and ramp change in surface temperature. Gladysz [3] solved the problem for a surface temperature changing as $t^2 \exp(-\alpha t)$ while White [4] solved it for surface heating which varied harmonically and Boley and Weiner [5] solve the problem for convection boundary conditions. Gladysz obtains a series solution, while the others are provided in closed form. Other studies have included the effects of coupling of the elastic and thermal equations, as well as heat deposition below the surface and thermal waves. Bushnell and McCloskey [6] obtained a closed form solutions for elastic waves due to volumetric heating, modeling the deposition using a non-zero attenuation coefficient for the heat incident on the surface of the solid. However, they ignored diffusion, assuming that the temperature profile matched the deposition profile. Similarly, Mozina and Dovc [7] and Galka and Wojnar [8] each assumed a volumetric heating given in the form $Q''' = a\mu \exp(-\mu x)$ where μ is the attenuation coefficient of the heat incident on the surface, but they included diffusion. All of these papers provide closed form solutions. Boley and Tolins [9] modeled the case with a step change in temperature, but included the coupling of the thermal and elastic equations by adding in a heating term which depended on the local strain rate. The solutions were obtained in terms of an integral and approximations were given in closed form for short and long times. Kao [10] solves the problem analytically for the non-Fourier case, in which the time scales of the heating are such that thermal waves are induced. Wang and Xu [11] included thermal waves, as well as volumetric heating and coupling of the thermal and elastic equations. Series solutions are obtained. These latter papers introduce undue complications for cases in which non-Fourier effects are insignificant, and are of questionable value since the hyperbolic equation solved has not been validated experimentally and has been shown to yield non-physical results for 3-D problems [12]. The time scales for which the solution derived in this paper is valid are discussed in the Results section of this paper.

Modeling

I begin by considering thermoelastic deformation of a half-space, with x denoting the perpendicular distance from the surface. Following Sternberg and Chakravorty [2], I define the following dimensionless variables

$$\xi = \frac{x}{a}$$
$$\tau = \frac{\kappa t}{a^2}$$
$$\phi = \frac{kT}{qa}$$
$$\hat{\sigma}_x = \frac{(1-2\nu)}{2(1+\nu)} \frac{k\sigma_x}{\alpha\, qa\mu} \qquad (1)$$

where $\hat{\sigma}$ is the dimensionless axial stress, ϕ is the dimensionless temperature, ξ and τ are the dimensionless coordinates for space and time respectively, κ is the thermal diffusivity, q is the surface heat load, α is the thermal expansion coefficient, μ is the shear modulus, ν is Poisson's ratio and

$$a = \frac{\kappa}{c}$$
$$c^2 = \frac{2(1-\nu)\mu}{(1-2\nu)\rho} \qquad (2)$$

Here c is the wave speed and ρ is the density of the solid.

With these definitions, the governing equations then become:

$$\frac{\partial^2 \phi}{\partial \xi^2} = \frac{\partial \phi}{\partial \tau}$$
$$\frac{\partial^2 \hat{\sigma}_x}{\partial \xi^2} = \frac{\partial^2 \hat{\sigma}_x}{\partial \tau^2} + \frac{\partial^2 \phi}{\partial \tau^2} \tag{3}$$

It is assumed here that the only non-zero displacement is perpendicular to the surface of the half-space. That is, u_y=u_z=0.The initial conditions are such that all temperatures, stresses, and time derivatives are nonexistent. The boundary conditions are that the temperatures and stresses vanish at x equals infinity, while at the surface

$$q = -k\frac{\partial T}{\partial x}$$
$$\sigma_x = 0 \tag{4}$$

which implies

$$\frac{\partial \phi}{\partial \xi} = -1$$
$$\hat{\sigma}_x = 0 \tag{5}$$

The solution for the dimensionless temperature is given by [13]

$$\phi = 2\left(\sqrt{\frac{\tau}{\pi}}\exp\left[\frac{-\xi^2}{4\tau}\right] - \frac{\xi}{2}erfc\left[\frac{\xi}{2\sqrt{\tau}}\right]\right) \tag{6}$$

and it's second derivative can thus be found to be

$$\frac{\partial^2 \phi}{\partial \tau^2} = \exp\left[\frac{-\xi^2}{4\tau}\right]\frac{\left(\xi^2 - 2\tau\right)}{4\sqrt{\pi}\,\tau^{5/2}} \tag{7}$$

The Laplace transform of this function is [14; p. 246, #15]

$$\overline{\phi} = \sqrt{s}\,\exp\left(-\xi\sqrt{s}\right) \tag{8}$$

Where s is the Laplace parameter and the bar over the function is meant to denote the transformed instance of the time function. Taking the transform of Eq. (3) and using the fact that all the initial values of the stress and its time derivatives are 0 gives

$$\frac{d^2\bar{\sigma}_x}{d\xi^2} - s^2\bar{\sigma}_x = \sqrt{s}\,\exp\left(-\xi\sqrt{s}\right) \tag{9}$$

Solving this equation and using the stress free boundary conditions yields

$$\bar{\sigma}_x = \frac{\exp\left(-\xi\sqrt{s}\right)}{(1-s)\sqrt{s}} - \frac{\exp(-\xi\, s)}{(1-s)\sqrt{s}} = \bar{\sigma}_1 + \bar{\sigma}_2 \tag{10}$$

Inverting the first term in this function requires some manipulation. I begin by letting $p = \sqrt{s}$ giving us

$$\bar{f} = \frac{\exp(-\xi\, p)}{p\left(1 - p^2\right)} \tag{11}$$

The inverse of this function is [14; p. 183, #22 and p. 170, #15]

$$f(\tau) = \left[1 - \cosh(\tau - \xi)\right]H(\tau - \xi) \tag{12}$$

where H(z) is the Heaviside step function. Given this result for f(τ), one can then find the time dependence of the original function using [14; p. 171, #29]

$$\hat{\sigma}_1 = \frac{1}{2\sqrt{\pi}\, t^{3/2}} \int_{\xi}^{\infty} u \exp\left(\frac{-u^2}{4\tau}\right)\left[1 - \cosh(u - \xi)\right]du \tag{13}$$

where the lower limit on the integral has been changed to reflect the step function in f(τ). Carrying out this integral and inverting the second term [14; p. 221, #1 and p. 170, #15] gives:

$$\hat{\sigma}_x = -\frac{1}{2}\exp(\tau - \xi)\left\{1 - \exp(2\xi)\, erfc\left(\frac{2\tau + \xi}{2\sqrt{\tau}}\right) + erf\left(\frac{2\tau - \xi}{2\sqrt{\tau}}\right) - 2\, erf\left(\sqrt{\tau - \xi}\right)H(\tau - \xi)\right\} \tag{14}$$

Having obtained this stress, we can obtain the remaining stresses by taking advantage of the assumption that there is no displacement parallel to the surface. Defining two more dimensionless stresses as

$$\hat{\sigma}_y = \frac{(1-2\nu)(1-\nu)}{2(1+\nu)}\frac{k\sigma_y}{\alpha q a \mu}$$
$$\hat{\sigma}_z = \frac{(1-2\nu)(1-\nu)}{2(1+\nu)}\frac{k\sigma_z}{\alpha q a \mu} \tag{15}$$

we obtain

$$\hat{\sigma}_y = \hat{\sigma}_z = \nu\hat{\sigma}_x - (1-2\nu)\phi \tag{16}$$

Substituting Eq. (14) into this expression gives us a solution for the two remaining normal stresses. This completes our solution for the stresses induced by surface heating on a half-space.

Since the longitudinal stress ($\hat{\sigma}_x$) is zero at the surface, the transverse stress ($\hat{\sigma}_y$) at the surface is given by

$$\hat{\sigma}_y = \hat{\sigma}_z = -(1-2\nu)\phi \tag{17}$$

or

$$\hat{\sigma}_y = \hat{\sigma}_z = -2(1-2\nu)\left(\sqrt{\frac{\tau}{\pi}}\right) \tag{18}$$

The peak stress in the wave occurs at $\xi=\tau$. Substituting this into Eq. (14) gives

$$\hat{\sigma}_x = -\frac{1}{2}\left\{1-\exp(2\tau)\,erfc\left(\frac{3\sqrt{\tau}}{2}\right)+erf\left(\frac{\sqrt{\tau}}{2}\right)\right\}$$
$$\hat{\sigma}_y = -\frac{\nu}{2}\left\{1-\exp(2\tau)\,erfc\left(\frac{3\sqrt{\tau}}{2}\right)+erf\left(\frac{\sqrt{\tau}}{2}\right)\right\}-2(1-2\nu)\left(\sqrt{\frac{\tau}{\pi}}\exp\left[\frac{-\tau}{4}\right]-\frac{\tau}{2}erfc\left[\frac{\sqrt{\tau}}{2}\right]\right) \tag{19}$$

For long times the longitudinal stress approaches –1, while the transverse stress approaches -ν. For short times, the leading terms for the longitudinal and transverse stresses are

$$\hat{\sigma}_x \approx 2\sqrt{\frac{\tau}{\pi}}$$
$$\hat{\sigma}_y \approx 2(1-\nu)\sqrt{\frac{\tau}{\pi}} \tag{20}$$

Results

Typical wave shapes are shown in Fig. 1, which plots the two dimensionless stresses as a function of distance from the surface. These results are given for dimensionless times of 0.5, 1, and 10. As one would expect, the stresses are all compressive, and the peak stress in the wave occurs at $\xi=\tau$. Except at early times, the transverse stress peaks at the surface because that's where the temperature peaks. At early times, there is a local peak in the transverse stress where the wave front lies, and at this point the transverse stress is less than the longitudinal stress.

Figure 2 displays the time dependence of the stresses at various depths from the surface. Each of the pairs of the curves is given at dimensionless times of 1 and 5. It is clear from this figure that the long-term behavior is dominated by the quasi-static stress, while the short term behavior is dominated by inertial effects.

The peak in the longitudinal stress occurs at $\xi=\tau$, while the peak transverse stress occurs at the surface (except at short times). These peaks are plotted in Fig. 3, which gives both stresses at $\xi=\tau$ along with the surface stress at the same dimensionless time. It can be seen that beyond a dimensionless time of approximately 4, the surface stress exceeds the stress at the wave peak. The point where the two are equal can be found more precisely by numerically solving Eqs. 18 and 19, giving a value of 3.81 for a Poisson's ratio of 0.3.

By comparing the surface stress to the peak stress in the wave, the results presented here are of interest for dimensionless times less than about 10. This corresponds to approximately 5 ps for aluminum and 25 ps for iron at room temperature. The pulse times of interest can also be much longer, because the stress wave will propagate much faster than the heat diffuses, and can cause spallation at a free surface at the back of a solid. In this case it will be the absolute magnitude of the peak stress in the wave, rather than its relation to the surface stress, that is of interest. On the other hand, the results are only valid for times long compared to the relaxation time associated with non-Fourier conduction. In most metals, this relaxation time is less than 0.01 picoseconds at room temperature and even smaller at higher temperatures. Hence there is a pulse length window from tens of fs to tens of ps (or greater) for which this solution is valid and meaningful.

Acknowledgements

This work was sponsored by the Naval Research Laboratory in support of the High Average Power Laser program.

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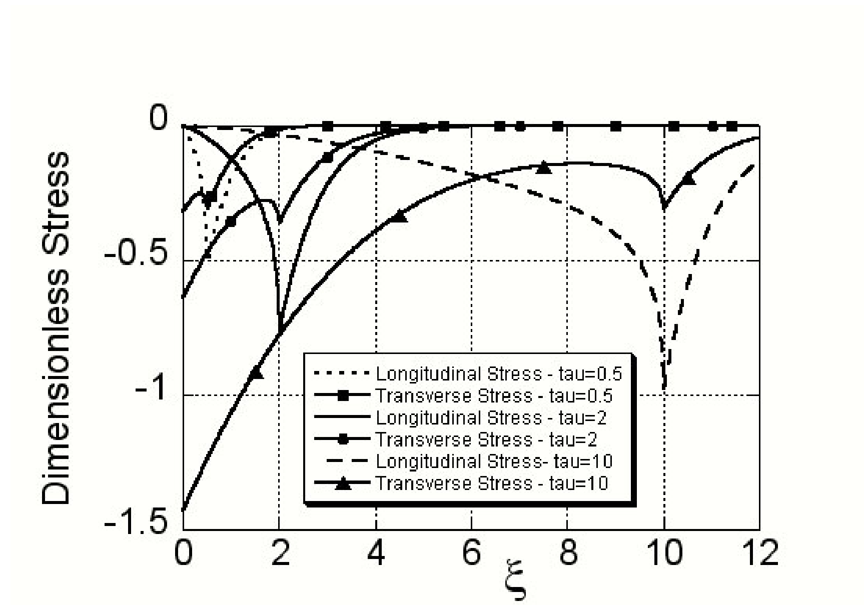
Figure Captions

Figure 1: Shape of stress fields at three different times. Transverse stresses assume ν=0.3.

Figure 2: Time dependence of stresses at two different locations relative to the surface. Transverse stresses assume ν=0.3.

Figure 3: Stresses vs. time at the surface and at the peak of the wave. Transverse stresses assume ν=0.3.



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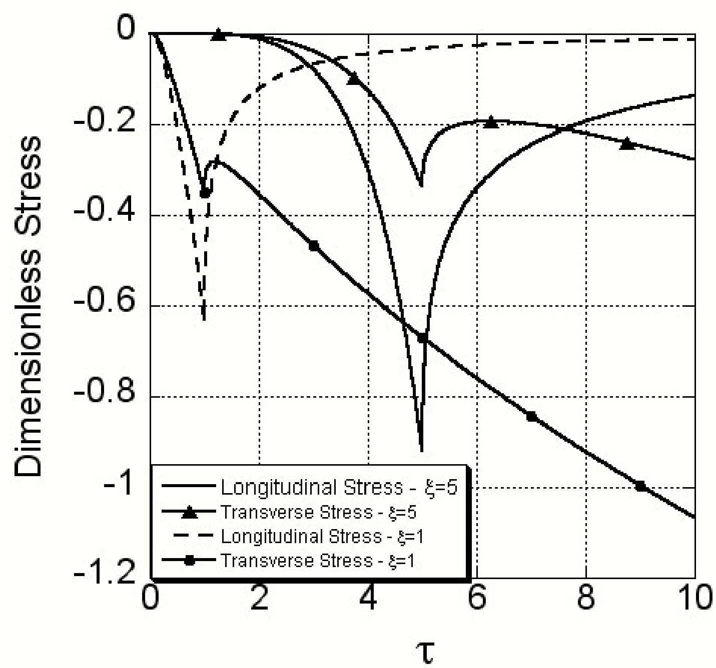
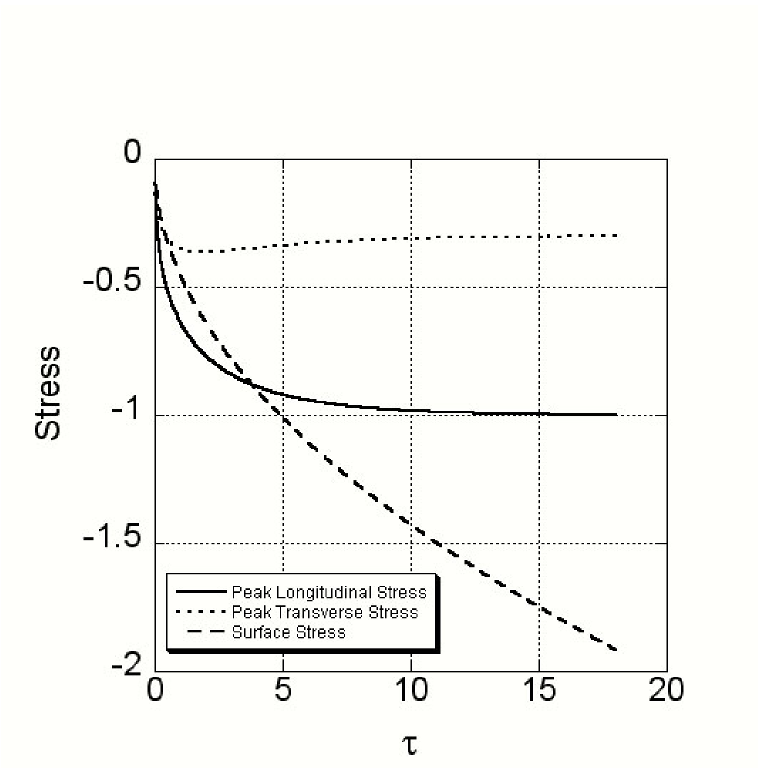


Figure 2: Time dependence of stresses at two different locations relative to the surface.

Transverse stresses assume ν=0.3.



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